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Almost Sure Convergence of the General Jamison Weighted Sum of $\mathcal{B}$-Valued Random Variables

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Abstract In this paper, two new functions are introduced to depict the Jamison weighted sum of random variables instead using the common methods, their properties and relationships are systematically discussed. We also analysed the implication of the conditions in previous papers. Then we apply these consequences to $\mathcal{B}$-valued random variables, and greatly improve the original results of the strong convergence of the general Jamison weighted sum. Furthermore, our discussions are useful to the corresponding questions of real-valued random variables.

Keywords Almost sure convergence, $\mathcal{B}$-valued random variable, General Jamison weighted sum, p-smooth Banach space, Banach space of type p

MR(2000) Subject Classification 60F15

1 Introduction

There has been much research work (for example, [1, 2] and [3]) about the almost sure convergence of the general Jamison weighted sum of real-valued independent random variables and negatively associated random variables, while the articles discussing the same problem in Banach space are very few. Recently, Liu Jingjun and Gan Shixin have done some significant work in this field (see [4]), but as compared with real-valued random variables, there still remains much to be desired. The purpose of this article is to make some progress in this situation. And the concepts in this article are the same as in [4].

In this paper, we let $\{\Omega, \mathcal{F}, \mathcal{P}\}$ be a complete probability space and $\mathcal{B}$ be a real separable Banach space with norm $\|\ \ \|$. The Banach space $\mathcal{B}$ is called type p $(1 \le p \le 2)$ if there exists a $c = c_p > 0$ such that

$$E\left\|\sum_{i=1}^{n} X_i\right\|^p \le c\sum_{i=1}^{n} E \| X_i \|^p, \quad n \ge 1,$$

where the independent $\mathcal{B}$-valued random variables $X_1, ..., X_n$ have mean zero and $E \| X_i \|^p < \infty,\ i = 1, ..., n$.

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Also in this paper, c denotes a finite positive constant which may be different at different places; $\{X_n\} \prec X$ means $\sup_n P(\| X_n \|> x) \leq cP(X > x)$, where $x > 0$ and X is some real-valued random variable.

Let $\{a_k, k \in \mathcal{N}\}$ and $\{b_k, k \in \mathcal{N}\}$ be sequences of real numbers, in which $a_k \neq 0$, $0 < b_k \uparrow \infty$. $\{X_n, n \in \mathcal{N}\}$ is a sequence of $\mathcal{B}$ -valued random variables. We will discuss the conditions satisfying

$$\lim_{n\to\infty} \frac{1}{b_n}\sum_{i=1}^{n} a_i X_i = 0 \quad \text{a.s.}$$

As compared with [4], there are several differences in our article:

1 We remove the requirement that $\{a_k, k \in \mathcal{N}\}$ is a sequence of positive numbers.

2 We remove the requirement that $\{b_k/|a_k|, k \in \mathcal{N}\}$ is strictly increasing.

3 We haven't any additional requirement about the convergence order of $\sum_{k=n}^{\infty} 1/\alpha_k^p$, where $\alpha_k = b_k/|a_k|, p > 0$.

First we introduce several notations:

$$\alpha_k = \frac{b_k}{|a_k|} \quad \text{for} \quad k \in \mathcal{N}, \qquad \text{and} \qquad N(x) = \#\{k : \alpha_k \leq x\} \quad \text{for} \quad x > 0,$$

where $\#A$ denotes the element number of set A, and we suppose $N(x) < \infty$, $\forall x > 0$.

Denote $x_0 = \inf\{x : N(x) > 0\}$. Clearly, from $N(1) < \infty$, we know that there are only finite elements in $\{k : \alpha_k \leq 1\}$. Hence $x_0 = \inf\{\alpha_k\} > 0$.

Now we define two other functions:

$$L(x) = \int_{x_0}^{x} \frac{N(t)}{t^2} dt = \int_{0}^{x} \frac{N(t)}{t^2} dt, \quad \text{and} \quad R_p(x) = \int_{x}^{\infty} \frac{N(t)}{t^{p+1}} dt,$$

for $x \geq x_0$ and $p > 0$. The function $N(x)$ is familiar, we can see it in many references (for example see [4]), but $L(x)$ and $R_p(x)$ are unfamiliar, here we need to introduce their background.

The following condition was used many times in [4]:

$$\max_{1\leq k\leq n} \alpha_k^p \sum_{k=n}^{\infty} \frac{1}{\alpha_k^p} = O(n), \tag{1.1}$$

where $p > 0$ and $\alpha_k = b_k/|a_k|$. Clearly, the necessary condition of (1.1) is

$$\sum_{k=1}^{\infty} \frac{1}{\alpha_k^p} < \infty. \tag{1.2}$$

First, we have the following

Lemma 1.1 *Condition* (1.2) *implies that*

$$R_p(x) < \infty, \qquad \forall x \geq x_0, \tag{1.3}$$

where p is the same as that in (1.2).

Proof First we have

$$\infty > \sum_{k=1}^{\infty} \frac{1}{\alpha_k^p} \geq \sum_{n=2}^{\infty} \sum_{k:n-1<\alpha_k\leq n} \frac{1}{\alpha_k^p} \geq \sum_{n=2}^{\infty} \frac{N(n) - N(n-1)}{n^p}$$

$$\geq 2^{-p}\sum_{n=2}^{\infty}\frac{N(n)-N(n-1)}{(n-1)^p} = 2^{-p}\sum_{n=2}^{\infty}N(n)\left(\frac{1}{(n-1)^p}-\frac{1}{n^p}\right)-2^{-p}N(1)$$
$$\geq \frac{p}{2^p}\sum_{n=2}^{\infty}N(n)\int_{n-1}^{n}\frac{1}{y^{p+1}}dy-2^{-p}N(1)\geq \frac{p}{2^p}\sum_{n=2}^{\infty}\int_{n-1}^{n}\frac{N(y)}{y^{p+1}}dy-2^{-p}N(1)$$
$$= \frac{p}{2^p}\int_{1}^{\infty}\frac{N(y)}{y^{p+1}}dy-2^{-p}N(1)=\frac{pR_p(1)}{2^p}-2^{-p}N(1).$$

Because $N(1)<\infty$, we have $R_p(1)<\infty$. Noting that $R_p(x)$ is a non-increasing function, $R_p(x)<\infty$, for all $x\geq 1$. Trivially, for $0<x_0<1$,

$$\int_{x_0}^{1}\frac{N(y)}{y^{p+1}}dy\leq\frac{N(1)}{x_0^{p+1}}<\infty,$$

which implies $R_p(x)<\infty$, for all $x\in(0,1)$.

This lemma explains the background of $R_p(x)$ very well. Moreover, we have

Lemma 1.2 *Suppose X is a non-negative real-valued random variable such that, for some $p>0$,*

$$EX^pR_p(X)<\infty. \tag{1.4}$$

Then we have

$$EX^rR_r(X)<\infty,\qquad \forall r>p, \tag{1.5}$$

and

$$EN(X)<\infty. \tag{1.6}$$

Proof For $x>x_0$, we have

$$x^rR_r(x)=x^r\int_x^{\infty}\frac{N(y)}{y^{r+1}}dy\leq x^p\int_x^{\infty}\frac{y^{r-p}N(y)}{y^{r+1}}dy=x^p\int_x^{\infty}\frac{N(y)}{y^{p+1}}dy=x^pR_p(x).$$

Hence (1.4) implies (1.5) for any $r>p$. Also

$$R_p(x)=\int_x^{\infty}\frac{N(y)}{y^{p+1}}dy\geq N(x)\int_x^{\infty}\frac{dy}{y^{p+1}}=\frac{1}{p}\frac{N(x)}{x^p},$$

so (1.4) implies (1.6).

For $L(x)$, obviously $L(x)<\infty,\forall x\geq x_0$, and if X is a non-negative real-valued random variable, we have the following lemma:

Lemma 1.3 *Suppose X is a non-negative real-valued random variable. Then the next two conditions are equivalent to each other:*

$$EXL(X)<\infty, \tag{1.7}$$
$$\int_1^{\infty}EN\left(\frac{X}{t}\right)dt<\infty, \tag{1.8}$$

and each of them implies that $EN(X/t)<\infty$, a.e. t.

Proof Obviously,

$$\int_1^{\infty}EN\left(\frac{X}{t}\right)dt=\int_1^{\infty}\left(\int_0^{\infty}N\left(\frac{x}{t}\right)dP(X\leq x)\right)dt$$

$$= \int_0^\infty \left(\int_1^\infty N\left(\frac{x}{t}\right) dt \right) dP(X \le x)$$
$$= \int_0^\infty x \left(\int_0^x \frac{N(y)}{y^2} dy \right) dP(X \le x)(\text{let } y = x/t)$$
$$= EXL(X).$$

So we see that (1.7) and (1.8) are equivalent to each other, and each of them implies $EN(X/t) < \infty$ a.e. t.

2 More Remarks on Condition (1.1)

In order to further our discussion and make comparison with [4], we will make more remarks on Condition (1.1). For this reason, we denote $d_n = \max_{1 \le k \le n} \alpha_k$. Clearly d_n is non-decreasing, and (1.1) is equivalent to

$$d_n^p \sum_{k=n}^\infty \frac{1}{\alpha_k^p} = O(n), \tag{2.1}$$

where $p > 0$. It is easy to see that (2.1) implies

$$\sum_{k=1}^\infty \frac{1}{\alpha_k^p} < \infty, \tag{2.2}$$

and

$$d_n^p \sum_{k=n}^\infty \frac{1}{d_k^p} = O(n). \tag{2.3}$$

(2.2) indicates that $\forall k \ge 1$, $\{\alpha_k\}$ is not a finite accumulative point, and $\alpha_k \to \infty$ as $k \to \infty$. Hence $N(x)$ is a right continuous ascending step function, and we can construct a continuous function $\widetilde{N}(x)$ that satisfies

$$\widetilde{N}(x) = \begin{cases} N(x) & \text{if } x \text{ is a jumping point of } N(x) \ , \\ \text{Linear} & \text{if } x \text{ lies between two jumping points of } N(x). \end{cases}$$

And let $\widetilde{L}(x) = \int_{x_0}^x \frac{\widetilde{N}(t)}{t^2} dt$, and $\widetilde{R_p}(x) = \int_x^\infty \frac{\widetilde{N}(t)}{t^{p+1}} dt$, for $x \ge x_0$ and $p > 0$. Clearly, $\widetilde{N}(x)$ and $\widetilde{L}(x)$ are strictly increasing, $\widetilde{R_p}(x)$ is strictly decreasing, and $N(x) \le \widetilde{N}(x), L(x) \le \widetilde{L}(x), R_p(x) \le \widetilde{R_p}(x)$.

Now we prove several lemmas.

Lemma 2.1 *Suppose there exists some $p > 0$ such that $R_p(x_0) < \infty$. Then $g_p(x) = x^p \widetilde{R_p}(x)$, $x \ge x_0$ must be a strictly increasing function of x.*

Proof By noting that $\widetilde{N}(x)$ is continuous, we have

$$g_p^{'}(x) = px^{p-1}\widetilde{R_p}(x) + x^p \frac{d\widetilde{R_p}(x)}{dx} = px^{p-1} \int_x^\infty \frac{\widetilde{N}(t)}{t^{p+1}} dt - \frac{1}{x}\widetilde{N}(x)$$
$$= \frac{1}{x}\left(px^p \int_x^\infty \frac{\widetilde{N}(t)}{t^{p+1}} dt - \widetilde{N}(x) \right) > \frac{1}{x}\left(px^p \widetilde{N}(x) \int_x^\infty \frac{1}{t^{p+1}} dt - \widetilde{N}(x) \right)$$
$$= \frac{1}{x}(\widetilde{N}(x) - \widetilde{N}(x)) = 0.$$

So $g_p(x)$ is strictly increasing when $x \ge x_0$.

Next we estimate the order of $N(d_n)$.

Lemma 2.2 *If* (1.1) *holds, then*

$$n \le N(d_n) \le cn, \tag{2.4}$$

where $c > 1$.

Proof Because $d_n = \max_{1\le k\le n} \alpha_k$, we have $\{1, 2, ..., n\} \subset \{k : \alpha_k \le d_n\}$. So $N(d_n) = \#\{k : \alpha_k \le d_n\} \ge n$. On the other hand, for $s > 0$, denoting

$$A_n(s) = \{\alpha_k : \alpha_k \le s, k \ge n\}, \quad B_n(s) = \{\alpha_k : \alpha_k \le s, 1 \le k < n\},$$

then by (2.1), there must exist $c_0 > 0$ such that

$$c_0 n \ge d_n^p \sum_{k=n}^{\infty} \frac{1}{\alpha_k^p} \ge d_n^p \sum_{k\in A_n(d_n)} \frac{1}{\alpha_k^p} \ge \#A_n(d_n).$$

Hence

$$N(d_n) = \#A_n(d_n) + \#B_n(d_n) \le c_0 n + n = (c_0 + 1)n \widehat{=} cn,$$

where $c > 1$.

Using Lemmas 2.1 and 2.2, we have:

Lemma 2.3 *Suppose X is a non-negative real-valued random variable. If* (1.1) *holds, then* (1.6) *is equivalent to*

$$EX^p R_p(X) \le EX^p \widetilde{R_p}(X) < \infty. \tag{2.5}$$

Proof By Lemma 1.2, we need to prove only that under Condition (1.1), the condition (1.6) implies (2.5).

From (2.1), we know that (2.3) and (2.4) hold. We notice that d_j, $j \ge 1$ must be the jumping points of $N(x)$, and $\widetilde{N}(x)$ is a strictly increasing function. $\forall n \in N$, we have

$$\begin{aligned}
\widetilde{R_p}(d_n) &= \int_{d_n}^{\infty} \frac{\widetilde{N}(y)}{y^{p+1}} dy = \sum_{k=n}^{\infty} \int_{d_n}^{d_{n+1}} \frac{\widetilde{N}(y)}{y^{p+1}} dy \le \frac{1}{p} \sum_{k=n}^{\infty} \widetilde{N}(d_{k+1}) \left[\frac{1}{d_k^p} - \frac{1}{d_{k+1}^p} \right] \\
&= \frac{1}{p} \sum_{k=n}^{\infty} N(d_{k+1}) \left[\frac{1}{d_k^p} - \frac{1}{d_{k+1}^p} \right] \qquad \text{(by definition of } \widetilde{N}(t)) \\
&\le c \sum_{k=n}^{\infty} k \left[\frac{1}{d_k^p} - \frac{1}{d_{k+1}^p} \right] = \frac{cn}{d_n^p} + c \sum_{k=n+1}^{\infty} \frac{1}{d_n^p}.
\end{aligned} \tag{2.6}$$

Noting that $g_p(x) = x^p \widetilde{R_p}(x)$ is a strictly increasing function, we have

$$\begin{aligned}
EX^p R_p(X) &\le EX^p \widetilde{R_p}(X) \le d_1^p \widetilde{R}(d_1) + \sum_{n=1}^{\infty} EX^p \widetilde{R_p}(X) I(d_n \le X < d_{n+1}) \\
&\le d_1^p \widetilde{R}(d_1) + \sum_{n=1}^{\infty} d_{n+1}^p \widetilde{R_p}(d_{n+1}) P(d_n \le X < d_{n+1}) \\
&\le d_1^p \widetilde{R}(d_1) + c \sum_{n=1}^{\infty} (n+1) P(d_n \le X < d_{n+1}) \\
&\quad + c \sum_{n=1}^{\infty} P(d_n \le X < d_{n+1}) d_{n+1}^p \sum_{k=n+1}^{\infty} \frac{1}{d_k^p}.
\end{aligned}$$

So by the condition (2.3), we have

$$\begin{aligned}EX^pR_p(X) &\le EX^p\widetilde{R_p}(X) \le d_1^p\widetilde{R}(d_1)+c\sum_{n=1}^{\infty}nP(d_n\le X<d_{n+1})\\&= d_1^p\widetilde{R}(d_1)+c\sum_{m=1}^{\infty}P(X\ge d_m)\le d_1^p\widetilde{R}(d_1)+c\sum_{m=1}^{\infty}P(N(X)\ge N(d_m))\\&\le d_1^p\widetilde{R}(d_1)+c\sum_{m=1}^{\infty}P(N(X)\ge m)=d_1^p\widetilde{R}(d_1)+cEN(X)<\infty.\end{aligned}$$

3 Preliminary Work

In this section, we prove several valuable lemmas. As compared with [2], our method is much concise.

Lemma 3.1 *If X is a non-negative real-valued random variable such that $EN(X)<\infty$, then $\sum_{k=1}^{\infty}P(X\ge\alpha_k)<\infty$.*

Proof We have

$$\begin{aligned}\sum_{k=1}^{\infty}P(X\ge\alpha_k)&=\sum_{k=1}^{\infty}\int_{\alpha_k}^{\infty}dP(X\le x)=\sum_{k=1}^{\infty}\int_0^{\infty}I(x\ge\alpha_k)dP(X\le x)\\&=\int_0^{\infty}\sum_{k=1}^{\infty}I(\alpha_k\le x)dP(X\le x)=\int_0^{\infty}N(x)dP(X\le x)\\&=EN(X)<\infty.\end{aligned}$$

Lemma 3.2 *If X is a non-negative real-valued random variable such that $E(X^pR_p(X))<\infty$, for some $0<p\le 2$, then*

$$\sum_{k=1}^{\infty}\frac{1}{\alpha_k^p}EX^pI(X\le\alpha_k)<\infty.$$

Proof First, from Lemma 1.2, we know

$$EN(X)\le pEX^pR_p(X)<\infty.$$

Because $\{\alpha_k\}$ is not necessarily monotone, we re-align α_1, α_2, ..., α_n as $\alpha_{n,1}\le\alpha_{n,2}\le\cdots\le\alpha_{n,n}$, if $\alpha_i=\alpha_j$ when $i<j$, then assume α_i precedes α_j. It is clear that $N(\alpha_{n,k})\ge k$, and using the inequality

$$EX^pI(X\le t)\le p\int_0^t x^{p-1}P(X>x)dx,$$

we have, for all $n\in\mathcal{N}$,

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{\alpha_k^p}EX^pI(X\le\alpha_k)&\le p\sum_{k=1}^{n}\frac{1}{\alpha_k^p}\int_0^{\alpha_k}x^{p-1}P(X>x)dx\\&=p\sum_{k=1}^{n}\int_0^1u^{p-1}P\Big(\frac{X}{u}>\alpha_k\Big)du=p\sum_{k=1}^{n}\int_0^1u^{p-1}P\Big(\frac{X}{u}>\alpha_{n,k}\Big)du\end{aligned}$$

$$\leq p\sum_{k=1}^{n}\int_0^1 u^{p-1}P\Big(N\Big(\frac{X}{u}\Big)\geq N(\alpha_{n,k})\Big)du \quad \text{(since } N(x) \text{ is non-decreasing)}$$
$$\leq p\sum_{k=1}^{n}\int_0^1 u^{p-1}P\Big(N\Big(\frac{X}{u}\Big)\geq k\Big)du \leq p\int_0^1 u^{p-1}\Big(\sum_{k=1}^{\infty}P\Big(N\Big(\frac{X}{u}\Big)\geq k\Big)\Big)du$$
$$= p\int_0^1 u^{p-1}EN\Big(\frac{X}{u}\Big)du = pE\int_0^1 u^{p-1}N\Big(\frac{X}{u}\Big)du$$
$$= p\int_0^{\infty}\Big(\int_0^1 u^{p-1}N\Big(\frac{x}{u}\Big)du\Big)dP(X\leq x)$$
$$= p\int_0^{\infty}x^p\Big(\int_x^{\infty}\frac{N(t)}{t^{p+1}}dt\Big)dP(X\leq x) = pE(X^pR_p(X)) < \infty.$$

The right hands of the above inequality is independent of n, so

$$\sum_{k=1}^{\infty}\frac{1}{\alpha_k^p}EX^pI(X\leq\alpha_k) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\alpha_k^p}EX^pI(X\leq\alpha_k) < \infty.$$

Lemma 3.3 *If X is a non-negative real-valued random variable such that $EXL(X) < \infty$, then*

$$\sum_{k=1}^{\infty}\frac{1}{\alpha_k}EXI(X\geq\alpha_k)\leq EXL(X) < \infty.$$

Proof Omitted.

Remark 3.1 The method of the proof is similiar to that of Lemma 3.2.

Lemma 3.4 *Suppose that X is a $\mathcal{B}$-valued random variable, and*

$$P(\|X\|\geq t)\leq cP(X_0\geq t),\ \forall t>0,$$

where X_0 is a non-negative real-valued random variable. Then $\forall q>0, t>0$, we have

$$E\|X\|^qI(\|X\|\leq t)\leq ct^qP(X_0>t)+cEX_0^qI(X_0\leq t),$$
$$E\|X\|^qI(\|X\|>t)\leq cEX_0^qI(X_0>t).$$

Proof Omitted.

4 Main Results

In this section, we suppose that X is a non-negative real-valued random variable, and $\mathcal{B}$ is a Banach space.

On the basis of the results in the above sections, we can improve Theorems 2.1–2.3 in [4] substantially.

Theorem 4.1 *Suppose that $\{a_k, k\in\mathcal{N}\}$ and $\{b_k, k\in\mathcal{N}\}$ are sequences of real numbers such that $a_k\neq 0$ and $0<b_k\uparrow\infty$. Let $\{X_n, n\in\mathcal{N}\}$ be a sequence of $\mathcal{B}$-valued random variables with $\{X_n\}\prec X$. If X satisfies*

$$EXR_1(X) < \infty, \tag{4.1}$$

then

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{k=1}^{n}a_kX_k = 0 \quad \text{a.s.} \tag{4.2}$$

Proof First, from Lemma 1.2, we know that (4.1) implies

$$EN(X) < \infty. \tag{4.3}$$

Define

$$Y_n = X_n I(\| X_n \| \le \alpha_n), \quad Z_n = X_n I(\| X_n \| > \alpha_n),$$

where $\alpha_n = b_n/|a_n|, n \in \mathcal{N}$. Noting that $\{X_n\} \prec X$, so by (4.3) and Lemma 3.1, we have

$$\sum_{n=1}^{\infty} P(\| Z_n \| \ne 0) = \sum_{n=1}^{\infty} P(X_n = Z_n) \le \sum_{n=1}^{\infty} P(\| X_n \| \ge \alpha_n) \le \sum_{n=1}^{\infty} P(X \ge \alpha_n) < \infty.$$

By the Borel–Cantelli lemma we get

$$P(\| Z_n \| \ne 0, \text{i.o.}) = 0.$$

Then

$$\left\| \sum_{n=1}^{\infty} \frac{a_n}{b_n} Z_n \right\| \le \sum_{n=1}^{\infty} \frac{\| Z_n \|}{\alpha_n} < \infty \qquad \text{a.s.}$$

Hence, by the Kronecher lemma we have

$$\frac{1}{b_n} \sum_{k=1}^{n} a_k Z_k \to 0 \qquad \text{a.s.} \qquad n \to \infty. \tag{4.4}$$

On the other hand, from Lemma 3.4 and $\{X_n\} \prec X$, we know

$$E \| Y_n \| = E \| X_n \| I(\| X_n \| \le \alpha_n) \le \alpha_n P(X > \alpha_n) + EXI(X \le \alpha_n).$$

Hence, by Lemmas 3.1 and 3.2 we have

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n} E \| Y_n \| \le \sum_{n=1}^{\infty} P(X > \alpha_n) + \sum_{n=1}^{\infty} \frac{1}{\alpha_n} EXI(X \le \alpha_n) < \infty.$$

So

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n} \| Y_n \| < \infty \qquad \text{a.s.}$$

By the Kronecker lemma again

$$\frac{1}{b_n} \sum_{k=1}^{n} |a_k| \, \| Y_k \| \to 0 \qquad \text{a.s.},$$

and using the C_r-inequality, we can get

$$\left\| \frac{1}{b_n} \sum_{k=1}^{n} a_k Y_k \right\| \le \frac{1}{b_n} \sum_{k=1}^{n} |a_k| \, \| Y_k \| .$$

So

$$\frac{1}{b_n} \sum_{k=1}^{n} a_k Y_k \to 0 \qquad \text{a.s.}, \qquad n \to \infty. \tag{4.5}$$

Then from (4.4) and (4.5), we obtain

$$\frac{1}{b_n} \sum_{k=1}^{n} a_k X_k \to 0 \qquad \text{a.s.}, \qquad n \to \infty.$$

Remark 4.1 Comparing with Theorem 2.1 in [4], we require neither $a_k > 0$ nor $\alpha_k = b_k/|a_k|$ to be strictly increasing, and Conditions (1) and (2) in Theorem 2.1 [4] implies (4.1), so our conditions are much weaker than those in Theorem 2.1 [4]. In addition, (4.1) is very concise.

Theorem 4.2 *Suppose that* $\{a_k, k \in \mathcal{N}\}$, $\{b_k, k \in \mathcal{N}\}$ *are the same as in Theorem* 4.1. *Let* $\mathcal{B}$ *be a* p*-smooth Banach space for some* $1 \le p \le 2$, *and* $\{X_n, n \in \mathcal{N}\}$ *be a sequence of* $\mathcal{B}$*-valued integrable random variables with* $\{X_n\} \prec X$. *If* X *satisfies*

$$E(X^p R_p(X)) < \infty, \tag{4.6}$$

$$EXL(X) < \infty, \tag{4.7}$$

then

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} a_k (X_k - E(X_k|\mathcal{F}_{k-1})) = 0 \qquad a.s.,$$

where $\mathcal{F}_0 = \{\phi, \Omega\}$, $\mathcal{F}_k = \sigma\{X_1, ..., X_k\}$, $k \ge 1$.

Proof First we define $\{Y_n\}$, $\{Z_n\}$ as in Theorem 4.1. To prove the result, we need to prove only that

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} a_k (Y_k - E(Y_k|\mathcal{F}_{k-1})) = 0 \qquad \text{a.s.}, \tag{4.8}$$

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} a_k (Z_k - E(Z_k|\mathcal{F}_{k-1})) = 0 \qquad \text{a.s.} \tag{4.9}$$

Noting that $\{\sum_{i=1}^m \frac{a_i}{b_i}(Y_i - E(Y_i|\mathcal{F}_{i-1})), \mathcal{F}_m, m \ge 1\}$ is a martingale, we need to prove only $\sum_{i=1}^\infty \frac{a_i}{b_i} U_i$ converges a.s. in order to prove (4.8), where $U_i := Y_i - E(Y_i|\mathcal{F}_{i-1}), i \ge 1$. And by the B-valued martingale convergence theorem [5] we need to prove only that $\{\sum_{k=1}^m \frac{a_k}{b_k} U_k, m \ge 1\}$ is L^p-bounded.

Using the property of a p-smooth Banach space, we have

$$\begin{aligned} E\left\|\sum_{k=1}^{m} \frac{a_k}{b_k} U_k\right\|^p &\le c \sum_{k=1}^{m} E\frac{1}{\alpha_k^p} \parallel U_k \parallel^p \le c_p 2^p \sum_{k=1}^{m} E\frac{1}{\alpha_k^p} \parallel Y_k \parallel^p \\ &\le c \sum_{k=1}^{m} \frac{1}{\alpha_k^p} E \parallel X_k \parallel^p I(\parallel X_k \parallel \le \alpha_k) \\ &\le c \sum_{k=1}^{\infty} \frac{1}{\alpha_k^p} E \parallel X_k \parallel^p I(\parallel X_k \parallel \le \alpha_k) \\ &\le c \sum_{k=1}^{\infty} P(X > \alpha_k) + c \sum_{k=1}^{\infty} \frac{1}{\alpha_k^p} EX^p I(X < \alpha_k). \end{aligned}$$

Hence, using the same method as in Theorem 4.1, we can get

$$\sup_{n\ge1} \left(E\left\|\sum_{k=1}^{m} \frac{a_k}{b_k} U_k\right\|^p\right)^{\frac{1}{p}} < \infty.$$

Now we prove (4.9). Let $V_k = Z_k - E(Z_k|\mathcal{F}_{k-1})$, $\forall k \ge 1$. Then

$$E \parallel V_k \parallel \le E \parallel Z_k \parallel + E \parallel E(Z_k|\mathcal{F}_{k-1} \parallel \le 2E \parallel Z_k \parallel, \quad \forall k \ge 1.$$

So by Lemma 3.3 and (4.7),

$$E\left(\left\|\sum_{k=1}^{\infty} \frac{a_k}{b_k} V_k\right\|\right) \le \sum_{k=1}^{\infty} \frac{1}{\alpha_k} E \parallel V_k \parallel \le 2 \sum_{k=1}^{\infty} \frac{1}{\alpha_k} E \parallel Z_k \parallel$$

$$\leq 2\sum_{k=1}^{\infty}\frac{1}{\alpha_k}\int_{\alpha_k}^{\infty}P(\| X_k \|\geq t)dt \leq 2\sum_{k=1}^{\infty}\frac{1}{\alpha_k}\int_{\alpha_k}^{\infty}cP(X\geq t)dt$$
$$\leq c\sum_{k=1}^{\infty}\frac{1}{\alpha_k}EXI(X\geq \alpha_k)\leq EXL(X)<\infty.$$

This implies

$$\left\|\sum_{k=1}^{\infty}\frac{a_k}{b_k}V_k\right\|<\infty \quad \text{a.s.}$$

Hence, by the Kronecher lemma again, (4.9) holds.

Remark 4.2 This is the same as Theorem 4.1, here we remove the requirement that $a_k > 0$ and $\alpha_k = b_k/|a_k|$ is strictly increasing. Conditions (1) and (3) in Theorem 2.2 [4] imply (4.6), and Condition (2) in Theorem 2.2 [4] is equivalent to (4.7) by Lemma 1.3.

Corollary 4.1 *Suppose that $\{a_n, n\geq 1\}$, $\{b_n, n\geq 1\}$ are the same as in Theorem 4.1. Let $\mathcal{B}$ be a p-smooth Banach space for some $1\leq p\leq 2$, and $\{X_n, \mathcal{F}_n, n\geq 1\}$ be a martingale difference sequence with $\{X_n\}\prec X$. If X satisfies $E(X^pR_p(X))<\infty$, and $EXL(X)<\infty$, then*

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{k=1}^{n}a_kX_k = 0 \quad a.s.$$

Theorem 4.3 *Suppose that $\{a_n, n\geq 1\}$, $\{b_n, n\geq 1\}$ are the same as in Theorem 4.1. Let $\mathcal{B}$ be of type p for some $1\leq p\leq 2$, and $\{X_n, n\geq 1\}$ be a sequence of $\mathcal{B}$-valued independent random variables with $EX_n = 0, n\geq 1$ and $\{X_n\}\prec X$. If X satisfies $E(X^pR_p(X))<\infty$, and $EXL(X)<\infty$, then*

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{k=1}^{n}a_kX_k = 0 \quad a.s.$$

Proof Omitted.

Remark 4.3 The method of the proof is similar to that of Theorem 4.2.

Corollary 4.2 *Suppose that $\{a_n, n\geq 1\}$, $\{b_n, n\geq 1\}$ are the same as in Theorem 4.1. Let $\mathcal{B}$ be of type p for some $1\leq p\leq 2$, and $\{X_n, n\geq 1\}$ be a sequence of $\mathcal{B}$-valued independent random variables with $\{X_n\}\prec X$. If X satisfies*

$$E(X^pR_p(X))<\infty,$$

then there exists $c_n\in\mathcal{B}$, $n = 1, 2, ...$, such that

$$b_n^{-1}\sum_{k=1}^{n}a_kX_k - c_n \to 0 \quad a.s.$$

Proof Define Y_n and Z_n as being the same as in the proof of Theorem 4.1. First it is easy to know that

$$b_n^{-1}\sum_{k=1}^{n}a_k(Y_k - EY_k)\to 0 \quad \text{a.s.}$$

On the other hand, $E(X^pR_p(X))<\infty$ implies $EN(X)<\infty$ by Lemma 1.2. So by Lemma 3.1

we can get

$$b_n^{-1}\sum_{k=1}^{n} a_k Z_k \to 0 \qquad \text{a.s.}$$

Then by letting $c_n = \frac{1}{b}_n \sum_{k=1}^{n} a_k EY_k$, we have

$$b_n^{-1}\sum_{k=1}^{n} a_k X_k - c_n \to 0 \qquad \text{a.s.}$$

The proof is finished.

Remark 4.4 Howell, Taylor and Woyczynshi [6] (1981) proved that, under the conditions $a_i > 0$ for $i \geq 1$, $EN(X) < \infty$ and

$$\int_0^\infty t^{p-1} P(X > t)\int_t^\infty \frac{N(y)}{y^{p+1}} dydt < \infty,$$

there exists $c_n \in \mathcal{B},\ n = 1, 2, ...$, such that

$$b_n^{-1}\sum_{k=1}^{n} a_k X_k - c_n \to 0 \qquad \text{a.s.}$$

Seeing that we have removed the conditions $EN(X) < \infty$ and $a_i > 0$ for $i \geq 1$, together with a trivial fact that

$$\int_0^\infty t^{p-1} P(X > t)\int_t^\infty \frac{N(y)}{y^{p+1}} dydt = \int_0^\infty t^{p-1} P(X > t) R_p(t) dt \geq EX^p R_p(X),$$

so we say that Corollary 4.2 improves their result.

Finally, as a supplement to Theorem 4.2, we will offer a result based on

$$EXL(X) = \infty. \tag{4.10}$$

Theorem 4.4 *Let $\{a_n, n \in \mathcal{N}\}$ and $\{b_n, n \in \mathcal{N}\}$ be sequences of positive numbers with $0 < b_k \uparrow \infty$ and*

$$\sum_{k=1}^{n} a_k = O(b_n), \quad n \to \infty. \tag{4.11}$$

Let $\mathcal{B}$ be of type p for some $1 \leq p \leq 2$, and $\{X_n, n \in N\}$ be a sequence of $\mathcal{B}$-valued independent random variables with $EX_n = 0$ and $\{X_n\} \prec X$. If X satisfies

$$EXL(X) = \infty, \quad E(X^p R_p(X)) < \infty, \quad EX < \infty, \tag{4.12}$$

then (4.2) *holds.*

Proof Using the same method as in Theorem 4.2, we can know that (4.4) and (4.5) hold, so it suffices to show that

$$\frac{1}{b_n}\sum_{k=1}^{n} a_k EY_k \to 0 \quad \text{a.s.} \tag{4.13}$$

Since $EX_k = 0, k \in N$, it suffices to show that

$$\frac{1}{b_n}\sum_{k=1}^{n} a_k EZ_k \to 0 \quad \text{a.s.} \tag{4.14}$$

Noting that a_k is non-negative, and

$$\begin{aligned} \| EZ_k \| &= \| EX_k I(\| X_k \| \geq \alpha_k) \| \\ &\leq E \| X_k \| I(\| X_k \| \geq \alpha_k) \leq cEXI(X \geq \alpha_k), \end{aligned}$$

hence, it suffices to show that

$$\frac{1}{b_n}\sum_{n=1}^{n} a_k EXI(X \geq \alpha_k) \to 0 \quad \text{a.s.} \tag{4.15}$$

By (4.11), there exists c_0 such that $0 < c_0 < \infty$, so that

$$\sum_{k=1}^{n} a_k \leq c_0 b_n, \tag{4.16}$$

for all $n \in \mathcal{N}$. From Lemma 1.2, we know that (4.6) implies

$$EN(X) < \infty.$$

So it is easy to know that $\alpha_k \to \infty$. $\forall \varepsilon > 0$, there exists $\beta > 0$ such that

$$EXI(X \geq \beta) < \frac{\varepsilon}{c_0},$$

where c_0 is the same as in (4.16). Since $\alpha_k \to \infty$, there exists $n_0 \in N$ such that

$$\min_{k>n_0} \alpha_k > \beta, \qquad \sup_{k>n_0} EXI(X \geq \alpha_k) < \frac{\varepsilon}{c_0}.$$

Hence, $\forall n > n_0$, we have

$$\frac{1}{b_n}\sum_{k=n_0+1}^{n} a_k EXI(X \geq \alpha_k) < \left(\frac{1}{b_n}\sum_{k=1}^{n} a_k\right)\frac{\varepsilon}{c_0} < \varepsilon; \tag{4.17}$$

on the other hand, clearly we have

$$\frac{1}{b_n}\sum_{k=1}^{n_0} a_k EXI(X \geq \alpha_k) \to 0, \quad n \to \infty.$$

So (4.14) holds.

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